RING HOMOMORPHISMS AND THE ISOMORPHISM THEOREMS

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ABSTRACT

In this presented paper ring Homeomorphisms and Isomorphism theorems, a branch of abstract algebra, a ring homomorphism is a structure-preserving function between two rings. More explicitly, if R and S are rings, then a ring homomorphism is a function f : R → S such that f is Additive inverses and the additive identity are part of the structure too, but it is not necessary to require explicitly that they too are respected, because these conditions are consequences of the three conditions above. If in addition f is bisection, then its inverse f^{-1} is also a ring homomorphism. In this case, f is called a ring isomorphism, and the rings R and S are called isomorphic. From the standpoint of ring theory, isomorphic rings cannot be distinguished. If R and S are rings, then the corresponding notion is that of a ring homomorphism,[b] defined as above except without the third condition $f(I_R) = I_S$. A ring homomorphism between (unital) rings need not be a ring homomorphism. The composition of two ring homeomorphisms is a ring homomorphism. It follows that the class of all rings forms a category with ring homeomorphisms as the morphisms (cf. the category of rings). In particular, one obtains the notions of ring endomorphism, ring isomorphism, and ring auto orphism.

Keywords: *Homomorphism, Isomorphism, algebra endomorphism, consequences*

Introduction

A map $f : R \to S$ between rings is called a **ring homomorphism** if
$f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R$.
The function θ:R → R/I defined by rθ = I +r is a ring homomorphism (called the canonical homomorphism for I) and its kernel is I. First Isomorphism Theorem for Rings If R and S are rings and φ:R → S is a ring homomorphism then R/ker(φ) ∼ = Im(φ). for all r1, r2 in R, so ψ is a homomorphism. A *homomorphism*is a map between two groups which respects the group structure. More formally, let G and H be two group, and f a map from G to H (for every g ∈G, f(g)∈H). Then f is a homomorphism if for every $g_1,g_2 \in G$, $f(g_1g_2)=f(g_1)f(g_2)$. For example, if H<G, then the inclusion map I (h)=h∈G is a homomorphism. Another example is a homomorphism from Z to Z given by multiplication by 2, f (n) =2n. This map is a homomorphism since f(n+m)=2(n+m)=2n+2m=f(n)+f(m).

Ring homomorphisms

Definition Let R and S be rings, and let φ:R → S be a function. Then φ is a ring homomorphism if

$(r1 + r_2)\varphi = r1\varphi + r_2\varphi$ and
$= (r1r_2)\varphi = (r1\varphi)(r_2\varphi)$ for all $r1, r_2$ in R.

Aring homomorphism which is a bijection is called an isomorphism. If there is an isomorphism φ from R to S then φ−1 is also an isomorphism and R is isomorphic to S, written R ~S
Definition If φ:R → S is a ring homomorphism then the image of φ is {rφ: r ∈R}, written Im(φ), and the kernel of φ is $\{r \in R : r\varphi = 0_S\}$, written ker(φ).

Theorem

If φ:R→S is a ring homomorphism then
(a) Im(φ) is a subring of S;
(b) ker(φ) is an ideal of R;
(c) r1 φ $=r_2\varphi$if and only if r1 and r_2 are in the same coset of ker(φ).

Proof

(a) We know that (Im (φ),+) is a subgroup of (S,+), from the similar theorem for groups. If s_1 and s_2 are in ker(φ) then there are r_1, r_2 in R with $r_1\varphi = s_1$ and $r_2\varphi = s_2$.
Then $s_1s_2 = (r_1\varphi)(r_2\varphi) = (r_1r_2)\varphi \in \ker(\varphi)$,
so Im(φ) ⩽ S.
(b) We know that (ker(φ),+) is a subgroup of (R,+), from the group theory. If r ∈ker(φ) andt ∈Rthen(rt)φ=(rφ)(tφ)=0S(tφ)=0S
and $(tr)\varphi=(t\varphi)(r\varphi)= (t\varphi)0_S = 0_S$.
Thus rt ∈ ker(φ) and tr ∈ ker(φ).
Therefore ker(φ)⊴R. 1
(c) We know this because φ:(R,+) → (S,+) is a group homomorphism.

The theorem has been stated in this way because parts (a) and (b) are so important. However, essentially the same proof can generalize part (a) to the image of any subring of R, and generalize part (b) to the inverse image of any subring or ideal of Im(φ)..

Theorem:
1) If φ:R→S is a homomorphism of rings, then the kernel of φ is an ideal of R, the image of φ is a subring of S and R/kerφ is isomorphic as a ring to φ(R).
2) If I is any ideal of R, then the map R→R/Idefined by r→r+I is a surjective ring homomorphism with kernel I. Thus, every ideal is the kernel of a ring homomorphism and vice versa.

Proof: Let φ:R→S be a ring homomorphism. Ifr∈R and r′∈kerφ, then wehave r∈kerφ (so that it is closed under multiplication by elements of R) since

φ(rr′)=φ(r)φ(r′)=φ(r)0=0=0φ(r)=φ(r′)φ(r) =φ(r′r);

since kerφ is also a subring of R, it is an ideal of R. It's clear that φ(R) is a subring of S. Now, let an ideal of R, so that R/I is also a ring, and define f:R→R/I by f(r)=r+I. We know f is a group homomorphism with kernelI, and for r,s∈R, we have

f(rs)=(rs)+I=(r+I)(s+I)=f(r)f(s),

so that f is in fact a ring homomorphism.

Define then ϕ:R/kerφ→φ(R)
Let kerφ=H
by
ϕ(r+(H))=φ(r),

for each (r+(H))∈R/H, for some r∈R. This is well defined because

if r′∈(r+(H)),, then

ϕ(r′+(H))=φ(r′)=φ(r)=ϕ(r+(H))
Also, this is a ring isomorphism because for each (s)∈φ(R) for some s∈R, we have

(∗) ϕ−1{φ(s)}=ϕ−1φ[r+(H)]=ϕ−1φ[f−1{r+(H)}]={r+(H)},

a set with a single element of R/H, so that it is a bijection, ϕ is injective, say
r+kerφ∈kerϕ i.e
0=ϕ (r+kerφ) = φ (r)
Which means r ∈kerφ, which then implies
r+kerφ=0+ker φ. So
ker ϕ={0+ker φ}
Similar that we can prove ϕ is surjective, since for every y ∈φ (R), their as exist r ∈ R, Such that
y= φ (r) = ϕ (r+ker φ)

and for every r+(H), for some r,r′∈R, we have
ϕ[(r+(H))+(r′+(H))]=ϕ[(r+r′)+(H)]=φ(r+r′) =φ(r)+φ(r′)=ϕ[r+(H)]+ϕ[r′+(H)],
and

ϕ[(r+(H))(r′+(H))]=ϕ[(rr′)+(H)]=φ(rr′)= φ(r) φ(r′)=ϕ[r+(H)]ϕ[r′+(H)],

so that it is a ring homomorphism.

The principal result is the following theorem: THEOREM. Let A and B be rings, a two-sided ideal of A, b a two-sided ideal of B and q : A + B a ring homomorphism inducing an isomorphism A/a z B/b. If p induces an isomorphism a/a" z b/b2 and an epimorphism q~* : Tort (A/a, A/a) + Torf (B/b, B/b), then v induces isomorphisms a"/a"+l G bn/b"fl and for n > 1. We then give some applications to supplemented algebras, graded algebras and free algebras.

Factor rings and the isomorphism theorems
We parallel the development of factor groups in Group theory.
Definition
If *I* is an ideal of a ring *R* and *a* ∈ *R* then a **coset** of *I* is a set of the form *a* + *I* = {*a* + *s* | *s I* }.
The set of all cosets is denoted by *R*/*I*.
Theorem
If I is an ideal of a ring R, the set R/I is a ring under the operations
(*a* + *I*) + (*b* + *I*) = (*a* + *b*) + *I* and (*a* + *I*) . (*b* + *I*) = (*ab*) + *I*.

Proof

We need to check that the operations are "well-defined". That is if a_1 and a_2 are representatives of the same coset and b_1 and b_2 represent the same coset then $a_1 + b_1$ and $a_2 + b_2$ represent the same coset and so do a_1b_1 and a_2b_2.

We have $a_1 - a_2 \in I$ and $b_1 - b_2 \in I$ and so adding these shows that $(a_1 + b_1) - (a_2 + b_2) \in I$ and so these do represent the same coset.

Similarly, for the product, observe that $a_1b_1 - a_2b_2 = (a_1 - a_2)b_1 + a_2(b_1 - b_2)$ and the result follows from the properties of the ideal.

Once you know that the operations are well-defined the ring axioms follow easily.

Note that the zero of the factor ring is the coset $0 + I$ or the ideal I itself.

Permutations and Group action

Since we introduced the definition of group as a set with a binary operation which is closed, we have been computing things internally, that is inside a group structure. This was the case even when considering Cartesian products of groups, where the first thing we did was to endow this set with a group structure.

In this section, we wonder what happens if we have a group and a set, which may or may not have a group structure. We will define a group action that is a way to do computations with two objects, one with a group law, not the other one.

Monomorphisms:

Injective ring homomorphisms are identical to monomorphisms in the category of rings: If $f : R \to S$ is a monomorphism that is not injective, then it sends some r_1 and r_2 to the same element of S. Consider the two maps g_1 and g_2 from Z[x] to R that map x to r_1 and r_2, respectively; $f \circ g_1$ and $f \circ g_2$ are identical, but since f is a monomorphism this is impossible.

However, surjective ring homomorphisms are vastly different from epimorphisms in the category of rings. For example, the inclusion $\mathbb{Z} \subseteq \mathbb{Q}$ is a ring epimorphism, but not a surjection. However, they are exactly the same as the strong epimorphisms.

Ring Homomorphism and Ideals:

In the study of groups, a homomorphism is a map that preserves the operation of the group. Similarly, a homomorphism between rings preserves the operations of addition and multiplication in the ring. More specifically, if R and S are rings, then a ring homomorphism is a

map $\phi:R\to S$ satisfying
$\phi(a+b)=\phi(a)+\phi(b)$ and $\phi(ab)=\phi(a)\phi(b)$

for all .a,b∈R. If $\phi:R\to S$ is a one-to-one and onto homomorphism, then ϕ is called an isomorphism of rings.

Example: For any integer n we can define a ring homomorphism $\phi:\mathbb{Z}\to\mathbb{Z}n$ by .$a\mapsto a(\mathrm{mod}\, n)$. This is indeed a ring homomorphism, since
$\phi(a+b)=(a+b)(\mathrm{mod}\, n)=a(\mathrm{mod}\, n)+b(\mathrm{mod}\, n)=\phi(a)+\phi(b)$
and
$\phi(ab)=ab(\mathrm{mod}\, n)=a(\mathrm{mod}\, n)\cdot b(\mathrm{mod}\, n)=\phi(a)\phi(b)$.
The kernel of the homomorphism ϕ is .

Statement of the theorems:-

Theorem A (groups):-

Let G and H be groups, and let $f: G \to H$ be a homomorphism. Then:

1. The kernel of f is a normal subgroup of G,
2. The image of f is a subgroup of H, and
3. The image of f is isomorphic to the quotient group $G / \ker(f)$.

In particular, if f is surjective then H is isomorphic to $G / \ker(f)$.

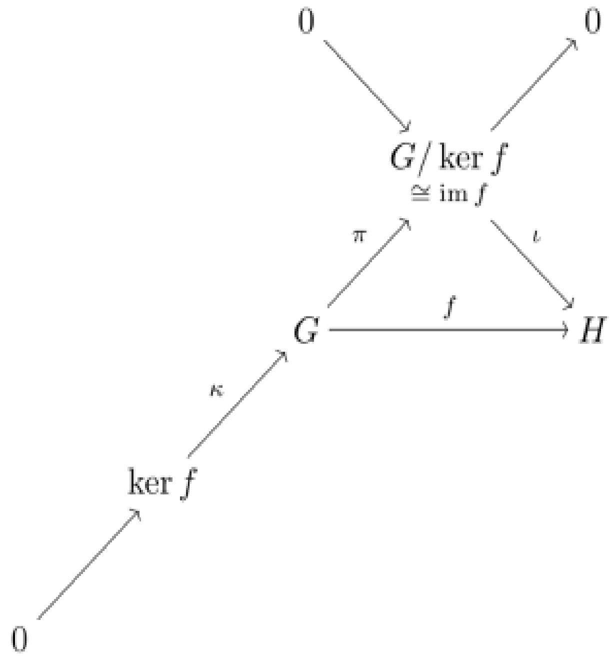

Diagram of the fundamental theorem on homomorphisms

Discussion

The first isomorphism theorem can be expressed in category theoretical language by saying that the category of groups is (normal epi, mono)-factorizable; in other words, the normal epimorphisms and the monomorphisms form a factorization system for the category. This is captured in the commutative diagram in the margin, which shows the objects and morphisms. Whose existence can be deduced from the morphism. The diagram shows that every morphism in the category of groups has a kernel in the category theoretical sense; the arbitrary morphism f factors into where ι is a monomorphism and f is an epimorphism (in a conormal category, all epimorphisms are normal). This is represented in the diagram by an object and a monomorphism are always monomorphisms), which complete the short exact sequence running from the lower left to the upper right of the diagram.

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